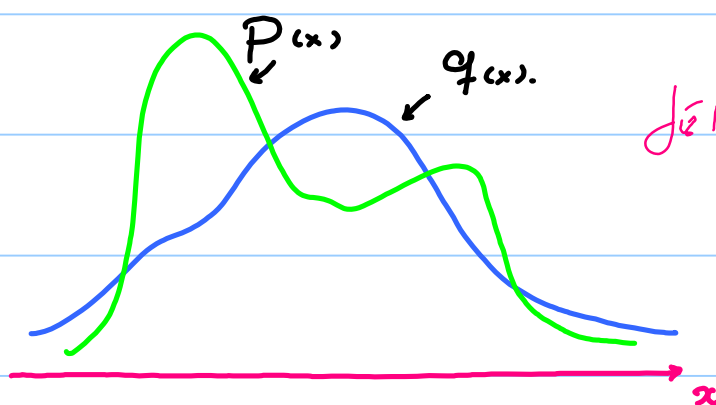


معیار فاصله بین حالت های پراش



● فاصله بین دو تابع توزیع احتمال

Def: $D(p, q) := \frac{1}{2} \sum_x |p_x - q_x|$

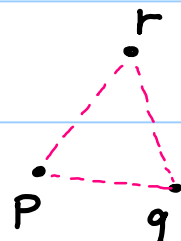
● خاصیت ۱:

● $D(p, q) \geq 0$

● $D(p, q) = 0 \iff p = q$

● $D(p, q) = D(q, p)$

● $D(p, q) \leq D(p, r) + D(r, q)$



• بهترین و با معیارترین حالت

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

$$S = \{x_1, x_3, x_7\}$$

$$P(S) = P(x_1) + P(x_3) + P(x_7)$$

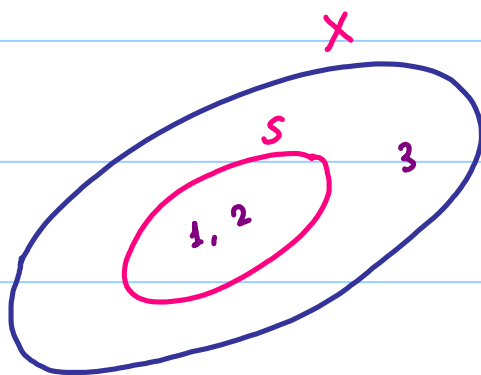
$$Q(S) = Q(x_1) + Q(x_3) + Q(x_7)$$

$$D(p, q) = \max_S |P(S) - Q(S)|$$

• قضیه:

• اثبات:

نمونه مثال:



$$|P(S) - Q(S)| = |P_1 + P_2 - Q_1 - Q_2| = |P_1 - Q_1 + P_2 - Q_2|$$

$$= \frac{1}{2} |P_1 - Q_1 + P_2 - Q_2| + \frac{1}{2} |P_1 - Q_1 + P_2 - Q_2|$$

$$= // + \frac{1}{2} |P_3 - Q_3|$$

$$\leq \frac{1}{2} |P_1 - q_1| + \frac{1}{2} |P_2 - q_2| + \frac{1}{2} |P_3 - q_3| = D(p, q).$$

$$|p(s) - q(s)| = \left| \sum_{x \in S} (P_x - q_x) \right| =$$

حالت کلی:

$$= \frac{1}{2} \left| \sum_{x \in S} (P_x - q_x) \right| + \frac{1}{2} \left| \sum_{x \in S} (P_x - q_x) \right|$$

$$= \frac{1}{2} \left| \sum_{x \in S} (P_x - q_x) \right| + \frac{1}{2} \left| \sum_{x \in S'} (P_x - q_x) \right|$$

$$\leq \frac{1}{2} \sum_{x \in X} |P_x - q_x| = D(p, q).$$

$$F(p, q) = \sum_x \sqrt{p_x q_x}$$

● شباهت بین دو توزیع احتمال

$$F(p, q) := \int dx \sqrt{p(x) q(x)}.$$

● Fidelity : نزدیکی

● $F(p, q) = F(q, p),$

- $0 \leq F(p, q) \leq 1$

- $F(p, q) = 1 \rightarrow p_\alpha = q_\alpha \quad \forall \alpha.$

Proof: $F(p, q) = \langle \tilde{p} | \tilde{q} \rangle$ where $\begin{cases} |\tilde{p}\rangle = (\sqrt{p_1}, \dots, \sqrt{p_n}) \\ |\tilde{q}\rangle = (\sqrt{q_1}, \dots, \sqrt{q_n}) \end{cases}$

Cauchy-Schwarz $\rightarrow \langle \tilde{p} | \tilde{q} \rangle^2 \leq \langle \tilde{p} | \tilde{p} \rangle \langle \tilde{q} | \tilde{q} \rangle = 1$

if $F(p, q) = 1 \rightarrow |\tilde{p}\rangle \parallel |\tilde{q}\rangle \rightarrow \text{Normalization} \Rightarrow |\tilde{p}\rangle = |\tilde{q}\rangle.$

فاصله بین حالت درکوانتومی
① اندازة هیلبرت

Hilbert-Schmidt

• اندازة هیلبرت - انجمن

$$\|A\|_{HS} = \|A\|_2 = \sqrt{\text{tr}(A^\dagger A)}$$

- $\|A\|_2 = 0 \rightarrow A = 0$

- $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$

- $\|cA\|_2 = |c| \|A\|_2, \quad \|A^\dagger\| = \|A^*\| = \|A\| = \|A^T\|$

- $\|U A U^\dagger\|_2 = \|A\|_2$

$$\|A^\dagger A\| \leq \|A\| \|A^\dagger\| = \|A\|^2$$

(۱) •

proof: $A^\dagger A$ is a positive, hermitian Matrix $=: X$

$$\|X\|^2 = \text{tr}(X^2) = \sum_i \lambda_i^2 \leq \left(\sum_i \lambda_i\right)^2 = (\text{tr} X)^2$$

$$\text{So} \rightarrow \|A^\dagger A\|^2 \leq \text{tr}(A^\dagger A)^2 = \|A\|^2.$$

$$\|AB\| \leq \|A\| \|B\|.$$

(۲) •

proof: $\|AB\|^2 = \text{tr}(AB B^\dagger A^\dagger) = \text{tr}(A^\dagger A B B^\dagger) =$

$$= \langle A^\dagger A | B B^\dagger \rangle$$

که در آن از ضرب داخلی هیلبرت استفاده کرده ایم.

$$\rightarrow \|AB\|^2 \leq \|A^\dagger A\| \|B B^\dagger\| \leq \|A\|^2 \|B\|^2$$

$$\|AB\| \leq \|A\| \|B\|.$$

سببرتی:

اندازه برد Trace Norm

$$\|A\|_1 := \text{tr} |A| \equiv \text{tr} \sqrt{A^\dagger A}$$

$$\bullet \quad \|A\|_1 = \sum_i |\lambda_i| \quad \lambda_i = \text{eigenvalue of } A$$

$$\text{proof: } A = \Omega D \Omega^\dagger, \quad A^\dagger = \Omega D^* \Omega^\dagger$$

$$AA^\dagger = \Omega D D^* \Omega^\dagger, \quad \sqrt{AA^\dagger} = \Omega \sqrt{D D^*} \Omega^\dagger$$

$$\text{tr} \sqrt{AA^\dagger} = \text{tr} \sqrt{D D^*} = \sum_i |\lambda_i|.$$

• $\|A\|_1$ مثبت است.

$$\bullet \quad \|A\|_1 = 0 \longrightarrow A = 0$$

$$\bullet \quad \|cA\|_1 = |c| \|A\|_1,$$

$$\bullet \quad \|A+B\|_1 \leq \|A\|_1 + \|B\|_1, \quad \text{proof?}$$

$$\bullet \quad \|A\|_1 = \|A^\dagger\|_1 = \|A^*\|_1 = \|A^\top\|_1;$$

- $\|UAU^\dagger\|_1 = \|A\|_1$

• فاصله بین دو حالت کوانتومی

$$D_1(\rho, \sigma) := \frac{1}{2} \text{tr} |\rho - \sigma|$$

• مثال:

$$\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma}) \quad \sigma = \frac{1}{2}(1 + \vec{s} \cdot \vec{\sigma})$$

$$D_1(\rho, \sigma) = \frac{1}{4} \text{tr} |(\vec{r} - \vec{s}) \cdot \vec{\sigma}|$$

eigenvalues of $\vec{r} \cdot \vec{\sigma} = R, -R \rightarrow D_1(\rho, \sigma) = \frac{1}{2} |\vec{r} - \vec{s}|$

$$\rho = \sum_i p_i |i\rangle\langle i|$$

• اگر $[\rho, \sigma] = 0$

$$\sigma = \sum_i q_i |i\rangle\langle i|$$

$$\rightarrow D_1(\rho, \sigma) = D_1(p, q)_{\text{classical}}$$

• فاصله کلاسیک

- $D_1(\rho, \sigma) = 0 \iff \rho = \sigma$

- $D_1(\rho, \sigma) \leq D_1(\rho, \mu) + D_1(\mu, \sigma)$

- $D_1(UpU^\dagger, U\sigma U^\dagger) = D_1(\rho, \sigma)$

• مهم ترین خاصیت D_1 .

$$D_1(\rho, \sigma) = \max_P \text{tr}(P(\rho - \sigma)).$$

این رابطه آنارک برابری رابطه مارکوف نیز است:

$$D(\rho, q) = \max_S |p(s) - q(s)|$$

• اثبات: $\rho - \sigma = Q - S$ هر دو بردار حالت:

که در آن Q, S ماتریس در مرتبه هستند.

$$\rho - \sigma = \begin{pmatrix} 1 & & & \\ & -2 & & \\ & & 5 & \\ & & & -4 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 5 & \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 0 & \\ & & & 4 \end{pmatrix} = Q - S$$

$$\text{tr} Q = \text{tr} S$$

$$|P-Q| = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 5 & \\ & & & 4 \end{pmatrix} = Q+S.$$

$$D(P, Q) = \frac{1}{2} \text{tr} |P-Q| = \frac{1}{2} \text{tr} (Q+S) = \text{tr} Q.$$

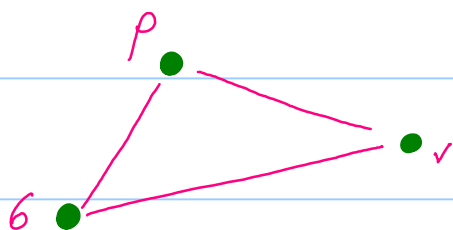
Let P be a projector. $P \geq 0 \rightarrow$

$$\text{tr}(P(P-Q)) = \text{tr}(P(Q-S)) \leq \text{tr}(PQ) \leq \text{tr}(Q)$$

So $\rightarrow \text{tr}(P(P-Q)) \leq D(P, Q).$

آیا می‌توانیم هر دو به هم برسانیم؟ پاسخ: نه.

then $\rightarrow \text{tr}(I_Q(P-Q)) = \text{tr}(Q) = D(P, Q).$



فاصله مثلث

$$D(P, Q) \leq D(P, 6) + D(6, Q)$$

• اثبات: با استفاده از قضیه قبل

$$D(\rho, \sigma) = \text{tr}(\rho^*(\rho - \sigma))$$

که در آن ρ^* مختار قرار است که به سبب قضیه قبل به آن تبدیل می‌کنند.

$$\begin{aligned} \rightarrow D(\rho, \sigma) &= \text{tr}(\rho^*(\rho - \nu) + \rho^*(\nu - \sigma)) \\ &= \text{tr}(\rho^*(\rho - \nu)) + \text{tr}(\rho^*(\nu - \sigma)) \\ &\leq D(\rho, \nu) + D(\nu, \sigma). \end{aligned}$$

Fidelity

• تشابه حالت و گرانتوی

$$F(\psi, \phi) := |\langle \psi | \phi \rangle|$$

$$F(\psi, \phi) = \sqrt{\langle \psi | \phi \rangle \langle \phi | \psi \rangle} = \sqrt{\text{tr}(\rho \sigma)}$$

The correct version: $F(\rho, \sigma) := \text{tr} \sqrt{(\sqrt{\rho} \sigma \sqrt{\rho})}$

- if ρ is pure $\rightarrow \rho = |\psi\rangle\langle\psi| \rightarrow \sqrt{\rho} = |\psi\rangle\langle\psi| \rightarrow$

$$(\sqrt{\rho} \sigma \sqrt{\rho}) = (|\psi\rangle\langle\psi| \sigma |\psi\rangle\langle\psi|) \rightarrow$$

$$\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} = \sqrt{\langle \psi | \sigma | \psi \rangle} |\psi\rangle\langle\psi| \rightarrow F(|\psi\rangle, \sigma) = \sqrt{\langle \psi | \sigma | \psi \rangle}$$

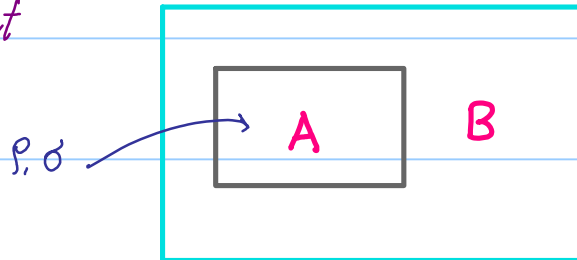
- if $[\rho, \sigma] = 0 \rightarrow \rho = \sum_i p_i |i\rangle\langle i|, \sigma = \sum_i q_i |i\rangle\langle i|$

$$\sqrt{\rho} \sigma \sqrt{\rho} = \rho \sigma = \sum_i p_i q_i |i\rangle\langle i|$$

$$\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} = \sum_i \sqrt{p_i q_i} |i\rangle\langle i|$$

$$F(\rho, \sigma) = \sum_i \sqrt{p_i q_i} = F(\rho, q)_{\text{class.}}$$

let



$$\rho = \text{tr}_B (|\psi\rangle\langle\psi|) \quad \sigma = \text{tr}_B (|\phi\rangle\langle\phi|)$$

$$F(\rho, \sigma) = \max_{\text{purifications}} |\langle\psi|\phi\rangle|$$

• یک کانال کوانتومی با نامیده حالت و شباهت آن چیست؟

• در اثر فزونی، شباهت در زیاد، تفاوت در کم می‌کند.

معنی: به اندازه حرکات $\mathcal{E}$

$$F(\rho, \sigma) \leq F(\mathcal{E}(\rho), \mathcal{E}(\sigma))$$

,

$$D(\rho, \sigma) \geq D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$$



ρ



σ



$\mathcal{E}(\rho)$



$\mathcal{E}(\sigma)$

مثال : $\mathcal{E}$ = Partial Trace

$$\mathcal{E}(\rho) = \text{tr}_B(\rho) \rightarrow$$



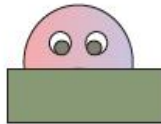
ρ



σ



$\mathcal{E}(\rho)$



$\mathcal{E}(\sigma)$

$$\rho - \sigma = Q - S$$

دليل :

اثبات : ●

$$Q, S \geq 0$$

$\downarrow$

$$\mathcal{E}(Q), \mathcal{E}(S) \geq 0$$

$$|\rho - \sigma| = Q + S$$

$$\text{tr}(\mathcal{E}(S)) = \text{tr}(S), \quad \text{tr}(\mathcal{E}(Q)) = \text{tr}(Q)$$

مخبر دليل

$$\rightarrow D(\rho, \sigma) = \frac{1}{2} \text{tr} |\rho - \sigma| = \text{tr}(\omega) = \text{tr}(\mathcal{E}(\omega))$$

$$\geq \text{tr}(\mathcal{E}(\omega) - \mathcal{E}(\sigma)) \quad (1) \quad \text{چون } \mathcal{E}(\sigma) \leq \mathcal{E}(\omega)$$

حال به این ترتیب با توجه قضیه قبلی می‌توانیم معرزش P_* را بدو برابر بنویسیم:

$$\text{tr}(P_*(\mathcal{E}(\omega) - \mathcal{E}(\sigma))) = \text{tr}(P_*(\mathcal{E}(\rho) - \mathcal{E}(\sigma))) \quad (2)$$

$$= D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$$

$$(1) \rightarrow D(\rho, \sigma) \geq \text{tr}(P_*(\mathcal{E}(\omega) - \mathcal{E}(\sigma))) = D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$$

$$D(\rho, \sigma) \geq D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$$

Distances Between Channels

Minimal Distance $D(\mathcal{E}_1, \mathcal{E}_2) = \frac{1}{2} \inf_p \| \mathcal{E}_1(\rho) - \mathcal{E}_2(\rho) \|_{\text{tr}}$

Maximal Distance $D_{\max}(\mathcal{E}_1, \mathcal{E}_2) = \frac{1}{2} \sup \|\mathcal{E}_1(\rho) - \mathcal{E}_2(\rho)\|_1$

Average Distance $D_{\text{av}}(\mathcal{E}_1, \mathcal{E}_2) = \int_{\mathcal{B}(\mathcal{H})} d\rho \|\mathcal{E}_1(\rho) - \mathcal{E}_2(\rho)\|_1$

$\phi_{\circ} :=$ Contraction to total mixture : disc

$\sigma_u :=$ unitary channel

$$\phi_{\circ}(\rho) = \frac{1}{d} I, \quad \sigma_u(\rho) = u \rho u^{\dagger}$$

$$D_{\min}(\sigma_u, \phi_{\circ}) = 0$$

$$D_{\max}(\sigma_u, \phi_{\circ}) = \frac{1}{2} \left(1 - \frac{1}{d} \right)$$

Quantification of Noise :

$$\Delta_{\text{sup}}(\mathcal{E}) := \frac{1}{2} \sup_{\rho} \|\mathcal{E}(\rho) - \rho\|$$

$C_{\psi} :=$ Contraction to pure state ψ

$$\Delta_{\text{sup}}(C_{\psi}) = \frac{1}{2} \sup_{\rho} \|\psi\psi^{\dagger} - \rho\|$$

$$\rho = \psi\psi^{\dagger} \rightarrow \Delta_{\text{sup}}(C_{\psi}) = 1.$$

Contraction to maximally mixed state

$$\Delta_{\text{sup}}(\phi) = \frac{1}{2} \sup_{\rho} \left\| \frac{1}{d} I - \rho \right\| = 1 - \frac{1}{d}$$

